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Wave Breaking in Beam-Plasma System with Finite Geometry

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We study wave breaking in a beam-plasma system placed in an infinite magnetic field, with finite geometry. A purely nonlinear nondispersive equation is deduced with the help of a reductive perturbation technique. Numerical analysis clearly shows that the initial profile of the wave (either parabolic or circular) grows with time leading to a discontinuous form of the wavefront, the phenomenon of wave breaking.

1. INTRODUCTION

Nonlinear wave propagation in a plasma is a problem of immense interest from both the theoretical and experimental standpoints (Lonngren, 1983). In this respect much work has been done in relation to the formation of the soliton (Lamb, 1980). On the other hand, an important class of events in a plasma is that of the breaking of nonlinear waves, which can take place for various reasons. Here we analyze this phenomenon in a beam-plasma system, taking account of the finite boundary condition (Das and Ghosh, 1986). It may be very much pertinent to point out that such a finite geometry is of utmost importance in actual experimental situations in a plasma.

2. FORMULATION

Here we consider the case of a hot plasma placed in an infinite magnetic field with the axis of the field pointing along the x-axis. We

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furthermore assume that the usual hydrodynamic description is possible. The governing equations can be written as

$$
\begin{aligned}
n_e^p \frac{\partial \phi}{\partial x} - \frac{\partial n_e^p}{\partial x} = 0 \\
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0 \\
\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} + \frac{\partial \phi}{\partial x} + \frac{3\sigma}{(1+\alpha)^2} n_i \frac{\partial n_i}{\partial x} = 0 \\
\frac{\partial n_e^b}{\partial t} + \frac{\partial}{\partial x}(n_e^b v_e^b) = 0 \\
\mu n_e^b \frac{\partial v_e^b}{\partial t} + \mu n_e^b v_e^b \frac{\partial v_e^b}{\partial x} - n_e^b \frac{\partial \phi}{\partial x} + \theta \frac{\partial n_e^b}{\partial x} = 0 \\
\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial x^2} = n_e^p + n_e^b - n_i
\end{aligned}
\tag{1}
$$

In the above equation n_e^p and n_e^b denote, respectively, the electron density in the plasma and the beam, and n_i that of the ions; v_e^b, v_e^p denote, respectively, the corresponding velocities of the two types of electrons and v_i that of the ions; ϕ denotes the electrostatic potential. We have normalized time to the inverse of the ion plasma frequency $\omega_i^p = (4\pi m_e e^2/m_i)^{1/2}$ where e is the charge of the electron and m_i is the mass of the ion. The densities have been normalized to n_{e_0}, the equilibrium electron density. The space coordinate x has been normalized to the electron Debye length $\lambda_D = (Te^p/4\pi m_{e_0} e^2)^{1/2}$, where T_e^p is the electron temperature. Furthermore, all the velocities have been normalized to the sound velocity of the plasma $C_s = (T_e^p/m_i)^{1/2}$ along with 0 to T_e^p/e. In the following the subscript zero indicates unperturbed value. Let us denote the ratio of the beam to plasma electron density as $\alpha = n_e^b/n_e^p$, $\theta = T_e^b/T_e^p$ and ion temperature $\sigma = T_i/T_e^p$, let μ denote the ratio m_e/m_i.

We start by defining the stretched variables (Washimi and Taniuti, 1966)

$$\xi = \epsilon(x - \lambda t), \qquad \tau = \epsilon^2 t \tag{2}$$

and the expansions

$$
\begin{aligned}
n_i &= n_i^0 + \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \cdots \\
v_i &= v_i^0 + \epsilon v_i^{(1)} + \epsilon^2 v_i^{(2)} + \cdots \\
n_e^b &= n_e^{b(0)} + \epsilon n_e^{b(1)} + \epsilon^2 n_e^{b(2)} + \cdots \\
v_e^b &= v_e^{b(0)} + \epsilon v_e^{b(1)} + \epsilon^2 v_e^{b(2)} + \cdots \\
\phi &= \phi + \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \cdots
\end{aligned}
\tag{3}
$$

are substituted in equations (1). Equating various power of ϵ, we obtain the following relations:

$$
\begin{aligned}
v_1^{(1)} &= \frac{\lambda - v_i^{(0)}}{(\lambda - v_i^{(0)})^2 - 3\sigma} \phi^{(1)} \\
n_i^{(1)} &= \frac{l + \alpha}{(\lambda - v_i^{(0)})^2 - 3\sigma} \phi^{(1)} \\
n_e^{b(1)} &= \frac{\alpha}{\lambda - v_0} v_e^{b(1)} \\
v_e^{b(1)} &= \frac{\lambda - v_0}{\theta - \mu(\lambda - v_0)^2} \phi^{(1)}
\end{aligned}
\tag{4}
$$

whereas the Laplace equation leads to

$$\frac{\partial^2 \phi^{(1)}}{\partial z^2} = q^2 \phi^{(1)} = 0 \tag{5}$$

$$q^2 = \frac{1+\alpha}{\lambda^2 - 3\sigma} - \frac{\alpha}{\theta - \mu(\lambda - v_0)^2} - 1 \tag{5a}$$

A simple solution of (5) is

$$\phi^{(1)} = f(\xi, \tau) \sin qz \tag{6}$$

On the boundary $z = b$, we must have $\phi^{(1)} = 0$ implying $q = n\pi/b$. Expressions (4) along with equation (6) yield the explicit form of all first-order quantities. Now in second order of ϵ we get

$$
\begin{aligned}
&\frac{1+\alpha}{\lambda^2 - 3\sigma^2} \sin qz \frac{\partial f}{\partial \tau} - \lambda \frac{\partial n_i^{(2)}}{\partial \xi} + (1+\alpha) \frac{\partial v_i^{(2)}}{\partial \xi} \\
&\quad + \frac{2\lambda(1+\alpha)}{(\lambda^2 - 3\sigma)^2} \sin^2 qz f \frac{\partial f}{\partial \xi} = 0
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&- \lambda \frac{\partial v_i^{(2)}}{\partial \xi} + \frac{\partial \phi^{(2)}}{\partial \xi} + \frac{3\sigma}{1-\alpha} \frac{\partial n_i^{(2)}}{\partial \xi} + \frac{\lambda^2}{(\lambda^2 - 3\sigma)^2} \sin^2 qz \\
&\quad \times f \frac{\partial f}{\partial \xi} + \frac{\lambda}{\lambda^2 - 3\sigma} \sin qz \frac{\partial f}{\partial \tau} + \frac{3\sigma}{(\lambda^2 - 3\sigma)^2} \sin^2 qz f \frac{\partial f}{\partial \xi} = 0
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&- (\lambda - v_0) \frac{\partial n_e^{b(2)}}{\partial \xi} + \alpha \frac{\partial v_e^{b(2)}}{\partial \xi} + \frac{\alpha}{\theta - \mu(\lambda - v_0)^2} \sin qz \frac{\partial f}{\partial \tau} \\
&\quad + \frac{\alpha(\lambda - v_0)}{[\theta - \mu(\lambda - v_0)^2]^2} \sin^2 qz f \frac{\partial f}{\partial \xi} = 0
\end{aligned}
\tag{9}
$$

$$- \mu\alpha(\alpha - v_0)\frac{\partial v_e^{b(2)}}{\partial \xi} - \alpha\frac{\partial \phi^{(2)}}{\partial \xi} + \theta\frac{\partial n_e^{b(2)}}{\partial \xi} + \sin^2 qx$$

$$\times \frac{-\alpha}{[\theta - \mu(\lambda - v_0)^2]^2} f\frac{\partial f}{\partial \xi} + \frac{\mu\alpha(\lambda - v_0)}{\theta - \mu(\lambda - v_0)^2}\sin qz\frac{\partial f}{\partial \tau} = 0 \quad (10)$$

$$\frac{\partial^2 \phi^{(2)}}{\partial z^2} = \phi^{(2)} + \frac{\phi^{(1)2}}{2} + n_e^{b(2)} - n_i^{(2)} \quad (11)$$

A simple manipulation of equations (7)–(11) leads to

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial \phi^{(2)}}{\partial \xi}\right) + q^2\frac{\partial \phi^{(2)}}{\partial \xi} + A\sin qz\frac{\partial f}{\partial t} + B\sin^2 qz f\frac{\partial f}{\partial \xi} = 0 \quad (12)$$

whence multiplying (12) by $\sin qz$ and integrating from 0 to b, we get

$$\frac{\partial f}{\partial \tau} + \left(\frac{B}{A}\int_0^b \sin qz\, dz\right) f\frac{\partial f}{\partial \xi} = 0 \quad (13)$$

where

$$A = \frac{2\mu\alpha(\lambda - v_0)}{[\theta - \mu(\lambda - v_0)^2]^2} + \frac{2\lambda(1+\alpha)}{(\lambda^2 - 3\sigma)^2}$$

and

$$B = \frac{2\mu\alpha(\lambda - v_0)^2}{[\theta - \mu(\lambda - v_0)^2]^3} - \frac{\lambda\mu\alpha(\lambda - v_0)}{[\theta - \mu(\lambda - v_0)^2]^3}$$
$$- \frac{\alpha}{[\theta - \mu(\lambda - v_0)^2]^2} + \frac{3\lambda^2(1+\alpha)}{(\lambda^2 - 3\sigma)^3}$$
$$+ \frac{3\sigma(1+\alpha)}{(\lambda^2 - 3\alpha)^3} - 1$$

On the other hand, from the condition that $q = n\pi/b$ and equation (5a) we get the following equation determining the phase velocity λ:

$$\lambda^4 - v_0\lambda^3 - C\lambda^2 + D\lambda + E = 0 \quad (14)$$

where

$$C = 3\sigma - v_0^2 + \frac{1+\alpha}{1 + n^2\pi^2/b^2} + \frac{\alpha}{\mu(1 + n^2\pi^2/b^2)}$$

$$D = 6\sigma^2 v_0 - \frac{2v_0(1+\alpha)}{1 + n^2\pi^2/b^2}$$

$$E = \frac{\theta(1+\alpha)}{\mu(1 + n^2\pi^2/b^2)} - 3\sigma v_0^2 - \frac{1+\alpha}{1 + n^2\pi^2/b^2}v_0^2 + \frac{3\sigma\alpha}{\mu(1 + n^2\pi^2/b^2)}$$

Equation (13) is the derived equation, which is seen to be dispersionless but nonlinear. Such an equation is also known as the Witham equation in the literature.

3. SOLUTION AND WAVE BREAKING

Since equation (13) is purely nonlinear and first order, it can be solved in the functional way. One can observe that the solution $f(\xi, \tau)$ can be written as

$$f = \phi(\xi - \alpha_1 f\tau) \tag{15}$$

where $\phi(\xi)$ denotes the initial form of the wave profile f. Here α_1 stands for $(B/A)\int_0^b \sin qz\, dz$.

Let us assume that ϕ is parabolic in shape. That is,

$$\phi = a^2 - \xi^2 \tag{16}$$

whence equation (15) can be explicitly solved for f and we get (Bhatnagar, 1982)

$$f(\xi, \tau) = \frac{(2\alpha_1\xi\tau - 1) + (4\alpha_1^2\tau^2a^2 + 1 - 4\alpha_1\xi\tau)^{1/2}}{2\alpha_1^2\tau^2} \tag{17}$$

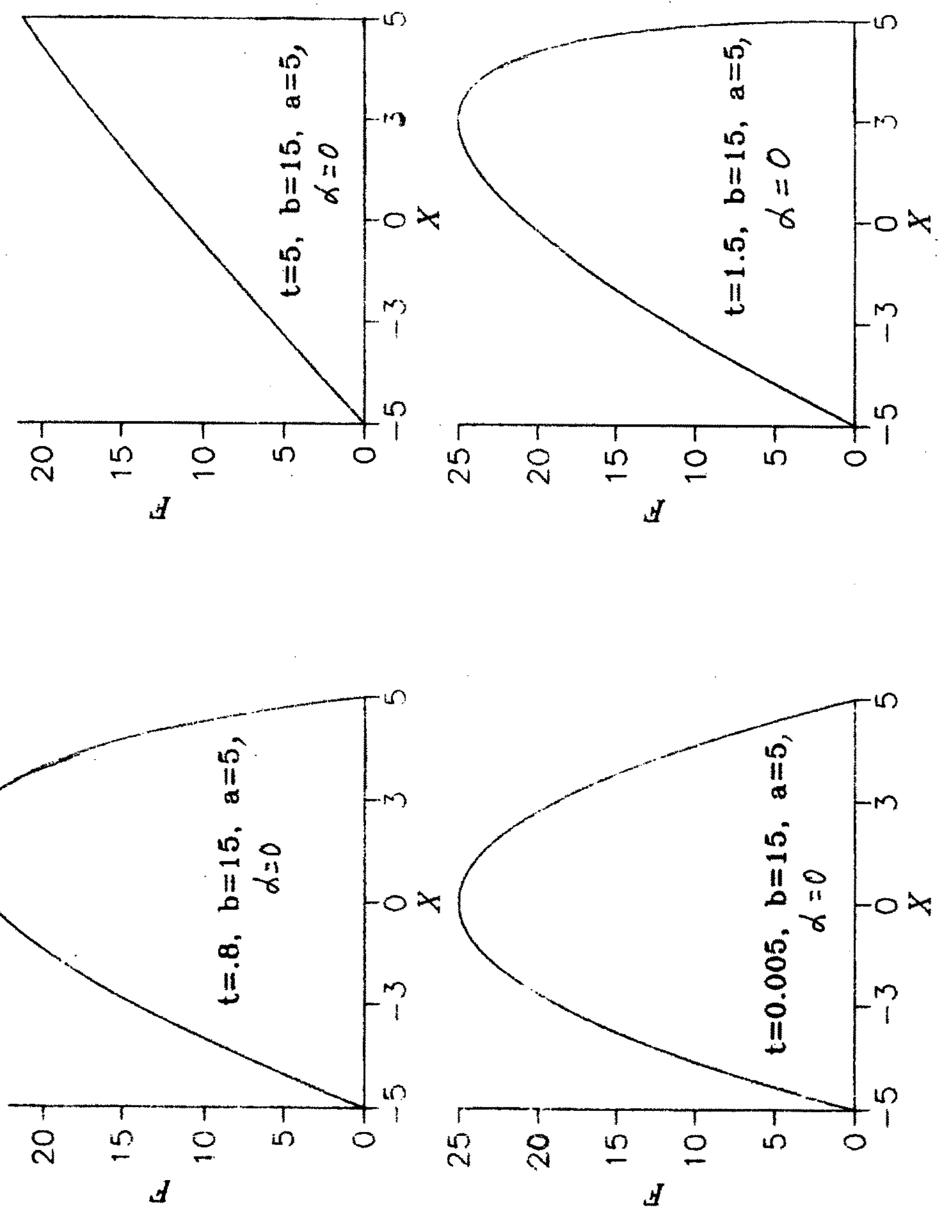
where the plus sign before the radical has been chosen from the consideration of the limiting value of f as $\tau \to 0$. On the other hand, if the initial wavefront is circular,

$$\phi^2 = a^2 - \xi^2 \tag{18}$$

then f turns out to be

$$f(\xi, \tau) = \frac{\alpha_1\xi\tau + (a^2 - \xi^2 + a^2\alpha_1^2\tau^2)^{1/2}}{1 + \alpha_1^2\tau^2} \tag{19}$$

To ascertain the behavior of the wavefront we have plotted numerically equations (17) and (19) for various values of the parameters in Figs. 1–3. For this purpose we required the values of the phase velocity λ to be determined from (14). We took $v_0 = 0.5$, $\theta = 0.1$, $\sigma = 0$, $\alpha = 0$ or 0.5, and $b = 0.8$ or 15 and determined λ from equation (14) and used (17) and (19) to study the evolution of the wave profile. It is clear from Figs. 1 and 2 that the parabolic wavefront degenerates into two pairs of straight lines and subsequently the wave breaks. This happens due to the excessive accumulation of energy in some particular mode and due to the absence of any dispersion. Figure 2 shows that for the small dimension of the confining system ($b < 1$) the breaking sets in earlier. So the inference is that the system becomes more unstable with reduction in b. Similarly, Fig. 3 shows


t=5, b=15, a=5, α=0
t=1.5, b=15, a=5, α=0
t=.8, b=15, a=5, α=0
t=0.005, b=15, a=5, α=0
F
X

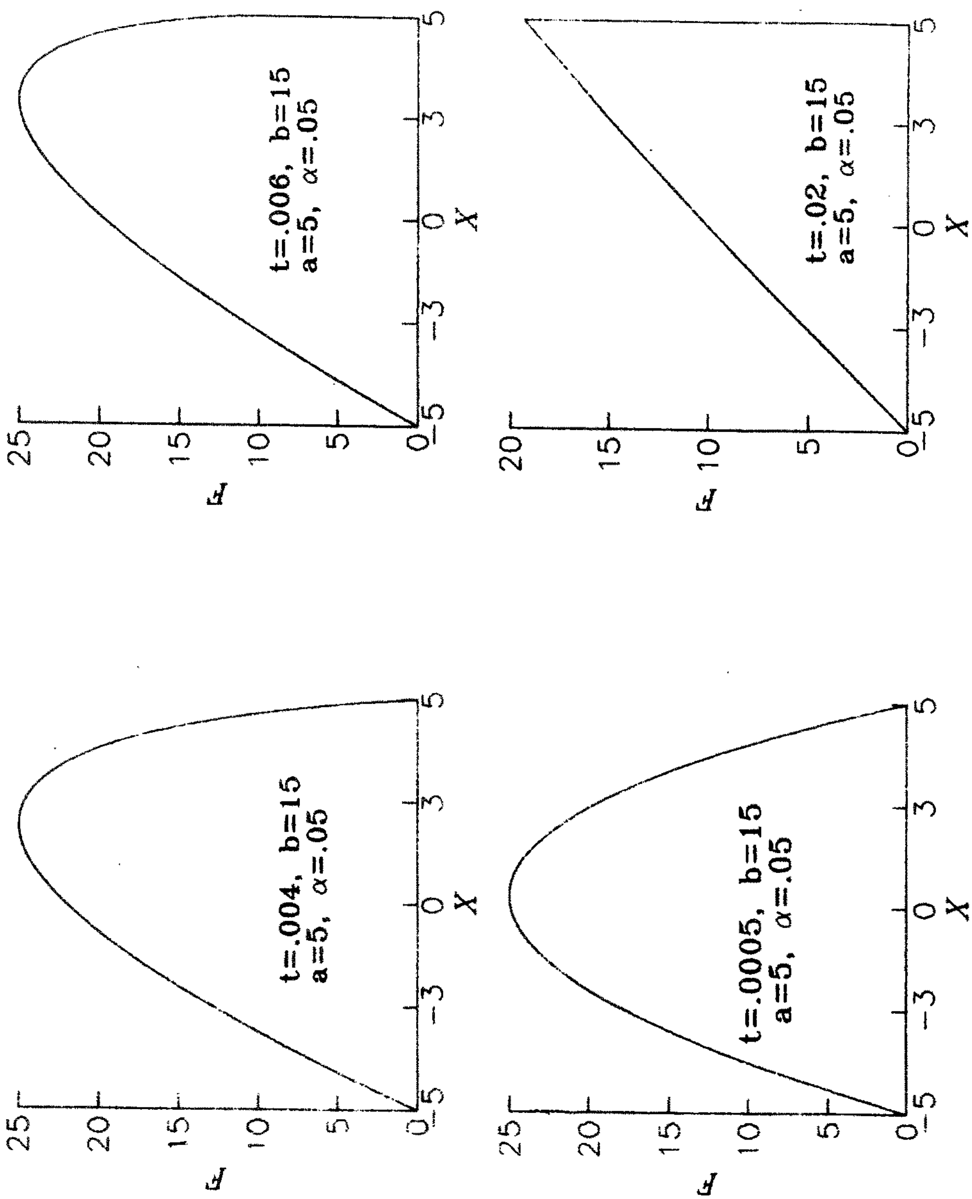
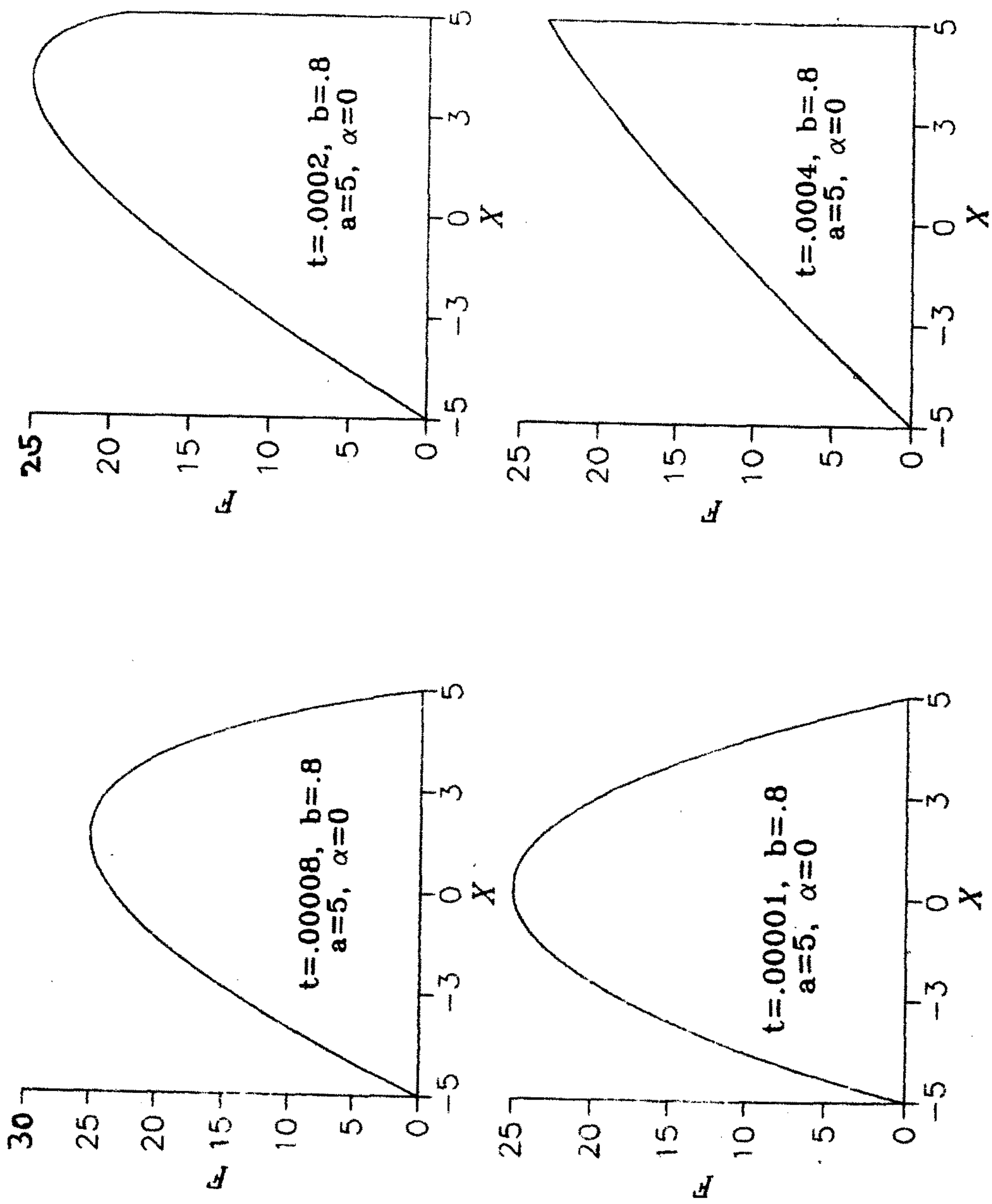


Fig. 1. Breaking in a parabolic wavefront for $b = 15$ and (a–d) $\alpha = 0$ and (e–f) $\alpha = .05$.


t=.0002, b=.8
a=5, α=0
t=.0004, b=.8
a=5, α=0
t=.00008, b=.8
a=5, α=0
t=.00001, b=.8
a=5, α=0
X
F

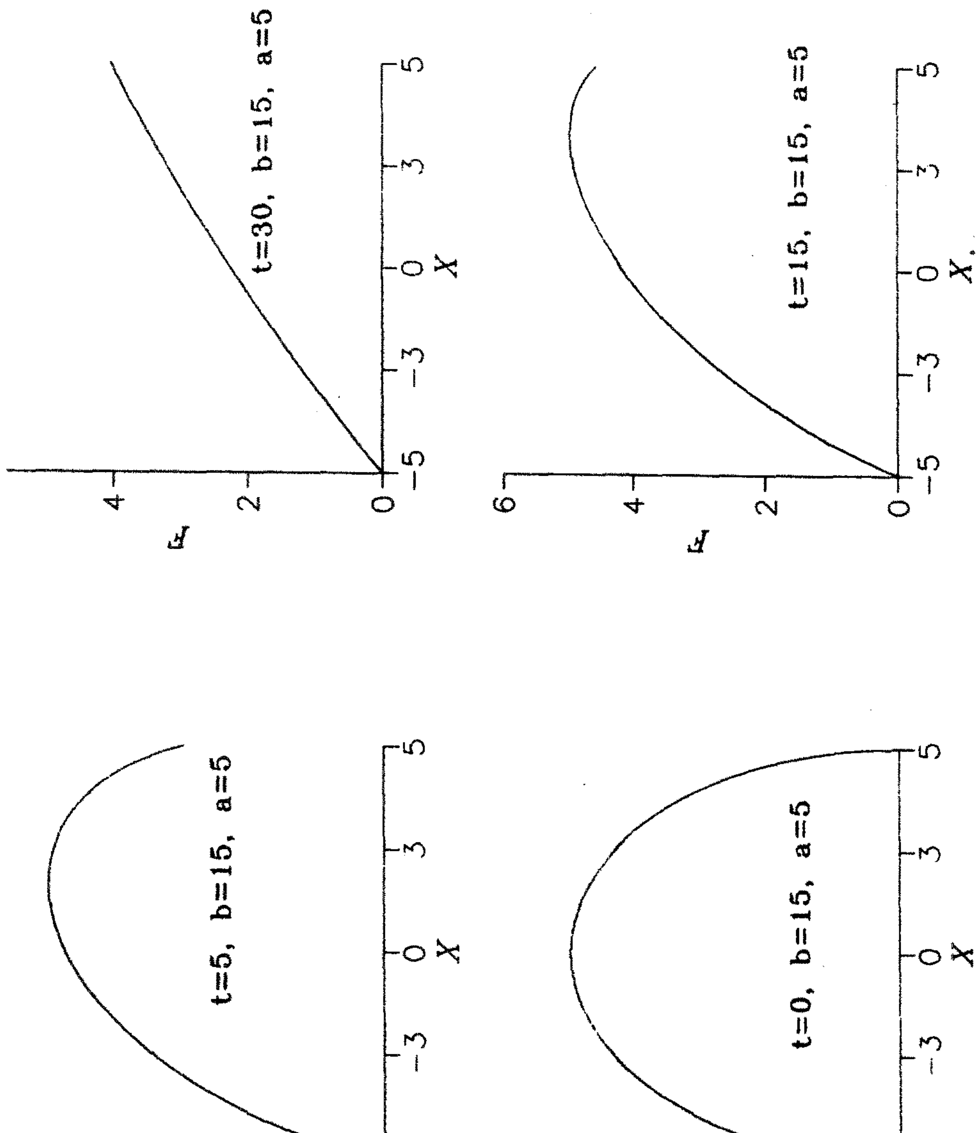


Fig. 3. Breaking in a circular wavefront for large values of *t*.

the situation when the wavefront at time $t = 0$ was circular. From this analysis it appears that one form of explosive instability is manifested in the phenomenon of wave breaking.

4. DISCUSSION

In the above analysis we showed that it is possible to study wave breaking in a plasma placed in an infinite magnetic field with finite boundaries, with the help of a reductive perturbation technique. Such phenomena are important for understanding explosive instabilities in various situations involving laser plasma interaction.

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